ภาวะต่อเนื่องบนปริภูมิ-q_s
The Continuity on q_s- Space

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บทคัดย่อ

วัตถุประสงค์ของการวิจัยในครั้งนี้เพื่อศึกษาภาวะต่อเนื่องบนปริภูมิ-q_s โดยอาศัยฟังก์ชันต่อเนื่อง -q_s และจากการศึกษาทำให้เราได้สมบัติที่สำคัญและหลากหลายซึ่งอยู่ในรูปของเซตปิด-q_s เซตเปิด-q_s เซตส่วนปิดคลุม -q_s เซตภายใน -q_s และเซตแก่นกลาง -q_s นอกจากนี้เรายังทำการศึกษาผลลัพธ์หรือสมบัติต่าง ๆ ที่เกิดขึ้นจากฟังก์ชันสมานสัณฐาน-q_s อีกด้วย

คำสำคัญ : ปริภูมิ -q_s , ฟังก์ชันต่อเนื่อง -q_s , ฟังก์ชันสมานสัณฐาน -q_s

Abstract

The purpose of this research is to investigate the continuity on q_s- space via q_s- continuous mapping. The findings found that the q_s- continuous mapping obtained various important properties in term of q_s- closed set, q_s- open set, q_s- closure $\left(c_{qs}\right)$ q_s- interior $\left(i_{qs}\right)$ and q_s- kernel$\left(k_{qs}\right)$. Furthermore, the study also examined the results in relation to q_s- homeomorphism properties.

Keywords: q_s- space, q_s- continuous mapping, q_s- homeomorphism

1. Introduction

Continuity was one of the main concepts for the most authors to study in any space especially, generalized topological spaces which is early focus space for them. Császár (Császár, 2002 a) introduced generalized open set and generalized continuity in generalized topology. In addition, Császár (Császár, 2012 b) was the first who introduced the concept of weak structure; it's one of the generalized topology. The weak structure was extended to generalized weak structure by Ávila (Ávila, 2012). After that, Janrongkam (Janrongkam, 2019) introduced the concept of quasi generalized weak structure. Later, Pongman (Pongman, 2019) introduce q-continuity and q^*-continuity of quasi generalized weak spaces and to give their equivalent forms. After that, Thongpan (Thongpan, 2019) extended quasi generalized spaces to the notion bi – quasi generalized weak spaces, In the last spaces, they were various variety of sets e.g. pq – interior sets, pq-closure and, Ladsungnern (Ladsungnern, 2020 a) introduced the concept pq-continuity and pq-homeomorphism and studied some important properties of them. Recently, Ladsungnern (Ladsungnern, 2021 b) extended from quasi generalized weak spaces to quasi generalized weak sum spaces (briefly, q_S - space) and studied the fundamental various properties of the set of q_S - interior, q_S -closure, q_S - kernel, q_S - exterior, q_S - boundary and q_S - derived in this space.

In this paper, the researcher introduced the concepts and studied some fundamental properties of q_S - continuity, q_S - open and q_S - closed mapping and q_S - homeomorphism on q_S - space.

2. Preliminaries

In this section, we presented to basic concepts of q_S - space and some fundamental properties . All results were presented by Ladsungnern (2021).

Definition 2.1 Let $\{(X_i,q_i):i\in I\}$ be any collection of pairwise disjoint quasi generalized weak spaces. Let $\bigoplus_{i\in I} X_i = \bigcup_{i\in I} X_i$ and q_S be a collection of subsets **of** $\bigoplus_{i\in I} X_i$ defined as follow: $q_S = \left\{U : U \subseteq \bigoplus_{i\in I} X_i \text{ and } U \cap X_i \in q_i\right\}$. The pair $\left(\bigoplus_{i\in I} X_i, q_s\right)$ is called the quasi generalized weak sum space of the space $\{(X_i,q_i):i\in I\}$ and briefly q_S - space. Each element of q_S is said to be q_S - open and the complement of q_S - open is called q_S - closed sets.

Example 2.2 Let $X_1=\{a,b\}$, $q_1=\{\{a\},\{a,b\}\}$,

$X_2=\{c,d,e\}$, $q_2=\{\{c\},\{c,d\},\{c,d,e\}\}$.

We obtained $I=\{1,2\}$, $\bigoplus_{i\in I} X_i=\{a,b,c,d,e\}$ and

$q_S = \{\{a,c\},\{a,b,c\},\{a,c,d\},\{a,c,d,e\},\{a,b,c,d\},\{a,b,c,d,e\}\}$.

The q_S - closed are $\varnothing,\{e\},\{b\},\{b,e\},\{d,e\},\{b,d,e\}$.

Throughout this paper, there are the symbols for use as follow:

1. $\left(\bigoplus_{i\in I} X_i, q_s\right)$ denoted by a q_S - space of the space $\{(X_i,q_i):i\in I\}$,

2. q_s - $O\left(\bigoplus_{i\in I} X_i\right)$ denoted by the set of all q_s - open sets on $\bigoplus_{i\in I} X_i$.

3. q_s - $C\left(\bigoplus_{i\in I} X_i\right)$ denoted by the set of all q_s - closed sets on $\bigoplus_{i\in I} X_i$.

Theorem 2.3 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A \subseteq \bigoplus_{i\in I} X_i$ be q_s - open set if $A = \bigcup_{i\in I}(A \cap X_i)$.

Note 2.4 1. If $G \in q_s$ - $O\left(\bigoplus_{i\in I} X_i\right)$ then $G = \bigcup_{i\in I}(G_i)$, when $G_i \in q_i$ - $O(X_i)$, $i \in I$.

2. If $F \in q_s$ - $C\left(\bigoplus_{i\in I} X_i\right)$ then $F = \bigcup_{i\in I}(F_i)$, when $F_i \in q_i$ - $C(X_i)$, $i \in I$.

Theorem 2.5 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and A, B are q_s - open sets. Then

1. $A \cup B$ is a q_s - open set,
2. $A \cap B$ is a q_s - open set.

Theorem 2.6 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be q_s - space , and A, B are q_s - closed sets. Then

1. $A \cup B$ is q_s - closed set,
2. $A \cap B$ is q_s - closed set.

Definition 2.7 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space and $A \subseteq \bigoplus_{i\in I} X_i$. The set of all q_s - interior of A denoted by $\mathrm{i}_{qs}(A)$, is defined by $\mathrm{i}_{qs}(A) = \bigcup\left\{U \in q_s\text{-}O\left(\bigoplus_{i\in I} X_i\right) : U \subseteq A\right\}$.

In other word, a point $x \in A$ is said to be q_s - interior point of A if there exist are q_s - open set U on $\bigoplus_{i\in I} X_i$ such that $x \in U \subseteq A$.

Theorem 2.8 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be q_s - a space $A, B \subseteq \bigoplus_{i\in I} X_i$. then

1. $\mathrm{i}_{qs}(\varnothing) = \varnothing$,
2. $\mathrm{i}_{qs}\left(\bigoplus_{i\in I} X_i\right) \subseteq \bigoplus_{i\in I} X_i$,
3. $\mathrm{i}_{qs}\, A \subseteq A$
4. If $A \subseteq B$ then $\mathrm{i}_{qs}(A) \subseteq \mathrm{i}_{qs}(B)$
5. $\mathrm{i}_{qs}(A)$ is a q_s --open set.
6. $\mathrm{i}_{qs}(A)$ is the largest q_s - open set contained in A.

Theorem 2.9 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A \subseteq \bigoplus_{i\in I} X_i$. Then

1. $A \in q_s\text{-}O\left(\bigoplus_{i\in I} X_i\right)$ if and only if $A = i_{qs}(A)$,
2. $\mathrm{i}_{qs}(\mathrm{i}_{qs}(A)) = \mathrm{i}_{qs}(A)$.

Theorem 2.10 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, $A \subseteq \left(\bigoplus_{i\in I} X_i\right)$. Then

$$\mathrm{i}_{qs}(A) \subseteq \bigcup_{i\in I} \mathrm{i}_{qi}(A \cap X_i).$$

Theorem 2.11 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A, B \subseteq \bigoplus_{i\in I} X_i$. Then

1. $\mathrm{i}_{qs}(A) \cup \mathrm{i}_{qs}(B) \subseteq \mathrm{i}_{qs}(A \cup B)$,
2. $\mathrm{i}_{qs}(A \cap B) = \mathrm{i}_{qs}(A) \cap \mathrm{i}_{qs}(B)$.

Definition 2.12 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A \subseteq \left(\bigoplus_{i\in I} X_i\right)$. The q_s - closure of A, denoted by $c_{qs}(A)$, is defined by $c_{qs}(A), = \cap\left\{F \in q_s\text{-}C\left(\bigoplus_{i\in I} X_i\right) : A \subseteq F\right\}$.

Theorem 2.13 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A, B \subseteq \bigoplus_{i\in I} X_i$. Then

1. $c_{qs}\left(\bigoplus_{i\in I} X_i\right)$,
2. $A \subseteq c_{qs}(A)$,
3. If $A \subseteq B$ then $c_{qs}(A) \subseteq c_{qs}(B)$,
4. $c_{qs}(A)$ is a q_s - closed set,
5. $c_{qs}(A)$ is the smallest q_s - closed set containing A,
6. $A \subseteq q_s\text{-}C\left(\bigoplus_{i\in I} X_i\right)$ if $c_{qs}(A) = A$.

Theorem 2.14 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A \subseteq \bigoplus_{i\in I} X_i$. Then

1. $[\mathrm{i}_{qs}(A)]^c - c_{qs}(A^c)$,
2. $[\mathrm{i}_{qs}(A^c)] = c_{qs}(A)^c$.

Theorem 2.15 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space, and $A \subseteq \bigoplus_{i\in I} X_i$. Then

$$\bigcup_{i\in I} c_{qi}(A \cap X_i) \subseteq c_{qs}(A).$$

Definition 2.16 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space and $A \subseteq \bigoplus_{i\in I} X_i$. The kernel q_s - of A denoted as K_{qs}, is defined by $K_{qs}(A) = \cap\left\{V : A \subseteq V \text{ and } V \in q_s\text{-}O\left(\bigoplus_{i\in I} X_i\right)\right\}$.

Note 2.17 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space and $A \subseteq \bigoplus_{i\in I} X_i$. Then

1. $K_{qs}(A) \in q_s\text{-}O\left(\bigoplus_{i\in I} X_i\right)$,
2. $\varnothing \subseteq K_{qs}(\varnothing)$,
3. $K_{qs}\left(\bigoplus_{i\in I} X_i\right) = \bigoplus_{i\in I} X_i$.

Theorem 2.18 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s -space and A, B are nonempty subset of $\bigoplus_{i\in I} X_i$. Then

1. $A \subseteq K_{qs}(A)$,
2. $A \in q_s\text{-}O\left(\bigoplus_{i\in I} X_i\right)$ then $A = K_{qs}(A)$,
3. $K_{qs}(K_{qs}(A)) = K_{qs}(A)$,
4. $A \subseteq B$ Then $K_{qs}(A) \subseteq K_{qs}(B)$,
5. $K_{qs}(A \cup B) = K_{qs}(A) \cup K_{qs}(B)$,
6. $K_{qs}(A \cap B) = K_{qs}(A) \cap K_{qs}(B)$.

Theorem 2.19 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ be a q_s - space and $A \subseteq \bigoplus_{i\in I} X_i$. Then

$\bigcup_{i\in I} K_{qi}(A \cap X_i) \subseteq K_{qs}(A)$.

3. Main Results

In this section, we discuss on q_s - continuity q_s - open and q_s - closed maps and q_s - homeomorphism on q_s - space.

3.1 q_s - continuity

Definition 3.1.20 Let $\left(\bigoplus_{i\in I} X_i, q_s\right)$ and $\left(\bigoplus_{j\in J} Y_j, q_s'\right)$ be a q_s - space. Then a mapping $f:\left(\bigoplus_{i\in I} X_i, q_s\right) \to \left(\bigoplus_{j\in J} Y_j, q_s'\right)$ is q_s - continuous on $\left(\bigoplus_{i\in I} X_i, q_s\right)$ if the inverse image of every q_s' - open set in $\left(\bigoplus_{j\in J} Y_j, q_s'\right)$ is q_s - open set in $\left(\bigoplus_{i\in I} X_i, q_s\right)$.

Example 3.1.21 Let

$I = \{1,2\}, X_1 = \{a,b\}, X_2 = \{c,d\}, q_1\{\{a\},\{a,b\}\}, q_2 = \{\{d\},\{c,d\}\}$.

$\bigoplus_{i\in I} X_i = X = \{a,b,c,d\}, J = \{3,4\}, Y_3 = \{e,k\}, Y_4 = \{g,h\}, q_3 = \{\{e\},\{e,k\}\}, q_4 = \{\{g\},\{g,h\}\}$.

$\bigoplus_{j\in J} Y_j = Y = \{e,k,g,h\}$,

$q_s = \{\{a,d\},\{a,c,d\},\{a,b,d\},\{a,b,c,d\}\}$

$q_s' = \{\{e,g\},\{e,g,h\},\{e,k,g\},\{e,k,g,h\}\}$, Let $f: X \to Y$ defined by $f(a)=e, f(b)=k, f(c)=h, f(d)=g$. Then $f^{-1}(\{e,g\})=\{a,d\}, f^{-1}(\{e,g,h\})=\{a,c,d\}, f^{-1}(\{e,k,g\})=\{a,b,d\}$ and $f^{-1}(\{e,k,g,h\})=\{a,b,c,d\}$. That is the inverse image of every q'_S - open set in Y is q_S - open set in X. Therefore, f is q_S - continuous.

Theorem 3.1.22 A mapping $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ is q_S - continuous if and only if the inverse image of every q'_S - closed set in $\underset{j\notin J}{\oplus} Y_j$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$.

Proof: Let f be q_S - continuous and F be q'_S - closed set in $\underset{j\in J}{\oplus} Y_j$. That is $\underset{j\in J}{\oplus} Y_j$ - F is q'_S - open in $\underset{j\in J}{\oplus} Y_j$ .Since f is q_S - continuous, $f^{-1}\left(\underset{j\in J}{\oplus} Y_j - F\right)$ is q_S - open in $\underset{i\in I}{\oplus} X_i$. That is , $\underset{i\in I}{\oplus} X_i - f^{-1}(F)$ is q_S - open in $\underset{i\in I}{\oplus} X_i$. Therefore, $f^{-1}(F)$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$.

Conversely, let the inverse image of every q'_S - closed in $\underset{j\in J}{\oplus} Y_j$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$. Let G be q'_S - open in $\underset{j\in J}{\oplus} Y_j$. Then $\underset{j\in J}{\oplus} Y_j$ - G is q'_S - closed set in $\underset{j\in J}{\oplus} Y_j$ .Then , $f^{-1}\left(\underset{j\in J}{\oplus} Y_j \text{ - } G\right)$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$. That is , $\underset{j\in J}{\oplus} Y_j$ - $f^{-1}(G)$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$ .Therefore, $f^{-1}(G)$ is q_S - open in $\underset{i\in I}{\oplus} X_i$. Thus, the inverse image of every q_S - open set in $\underset{j\in J}{\oplus} Y_j$ is q_S - open in $\underset{i\in I}{\oplus} X_i$.That is f is q_S - continuous on $\underset{i\in I}{\oplus} X_i$. #

Theorem 3.1.23 A mapping $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ is q_S - continuous if and only if $f\left(c_{qS}(A)\right) \subseteq c_{q'S}\left(f(A)\right)$ for every $A \subseteq \underset{i\in I}{\oplus} X_i$.

Proof Let f be q_S - continuous mapping and $A \subseteq \underset{i\in I}{\oplus} X_i$.Then $f(A) \subseteq \underset{j\subset J}{\oplus} Y_j$ so $c_{q'S}\left(f(A)\right)$ is q'_S -closed set in $\underset{j\in J}{\oplus} Y_j$ since f is q_S -continuous mapping, we have $f^{-1}\left(c_{q'S}\left(f(A)\right)\right)$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$. Since $f(A) \subseteq c_{q'S}\left(f(A)\right) \Rightarrow A \subseteq f^{-1}\left(c_{q'S}\left(f(A)\right)\right)$.Thus $f^{-1}\left(c_{q'S}\left(f(A)\right)\right)$ is a q_S - closed containing A .But, $c_{qS}(A)$ is the smallest q_S - closed set containing A. Therefore $c_{qS}(A)$ $\subseteq f^{-1}\left(c_{q'S}\left(f(A)\right)\right)$ .That is $f\left(c_{qS}(A)\right) \subseteq c_{q'S}\left(f(A)\right)$.

Conversely, let $f\left(c_{qS}(A)\right) \subseteq c_{q'S}\left(f(A)\right)$ for $A \subseteq \underset{i\in I}{\oplus} X_i$. If F is q'_S - closed in $\underset{j\in J}{\oplus} Y_j$,since $f^{-1}(F) \subseteq \underset{i\in I}{\oplus} X_i$, $f\left(c_{qS}\left(f^{-1}(F)\right)\right) \subseteq c_{q'S}\left(f\left(f^{-1}(F)\right)\right) \subseteq c_{q'S}(F)$.

That is $c_{qs}\left(f^{-1}(F)\right)\subseteq f^{-1}\left(c_{q's}(F)\right)=f^{-1}(F)$. But $f^{-1}(F)\subseteq c_{qs}\left(f^{-1}(F)\right)$.
Hence, $c_{qs}\left(f^{-1}(F)\right)=f^{-1}(F)$.Therefore, $f^{-1}(F)$ is q_S- closed in $\underset{i\in I}{\oplus}X_i$ for every q'_S- closed F in $\underset{j\in J}{\oplus}Y_j$. By Theorem 3.1.22, we obtained f is q_S- continuous mapping. #.

Remark 3.1.24 If $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is q_S- continuous, then $f\left(c_{qs}(A)\right)$ is not necessarily equal to $c_{q's}\left(f(A)\right)$ where $A\subseteq\underset{i\in I}{\oplus}X_i$. For example,

Let $I=\{1,2\},X_1=\{a,b\},X_2=\{c,d\},\underset{i\in I}{\oplus}X_i=\{a,b,c,d\}$,

$q_1=\{\{a\},\{a,b\}\},q_2=\{\{c\},\{c,d\}\}$,

$q_S=\{\{a,c\},\{a,c,d\},\{a,b,c\},\{a,b,c,d\}\}$,

$J=\{3,4,5\},Y_3=\{e,g\},Y_4=\{h\},Y_5=\{k,s\}$,

$q_3=\{\{e\}\},q_4=\{\{h\}\},q_5=\{\{k\},\{k,s\}\}$,

$q'_s=\{\{e,h,k\},\{e,h,k,s\}\}$.

Let $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ be defined by $f(a)=e,f(b)=s,f(c)=h,f(d)=k$.

We have $f^{-1}\{e,h,k\}=\{a,c,d\}\in q_S\text{-}O\left(\underset{i\in I}{\oplus}X_i\right),f^{-1}\{e,h,k,s\}$
$=\{a,b,c,d\}\in q_S\text{-}O\left(\underset{i\in I}{\oplus}X_i\right)$.

Hence f is q_S - continuous on $\underset{i\in I}{\oplus}X_i$.

Let $A=\{b\}\subseteq\underset{i\in I}{\oplus}X_i,c_{qs}(A)=\{b\},f\{b\}=\{s\}\Rightarrow f(A)=\{s\},c_{q's}\{s\}=\{k,s\}$.

Therefore, $f\left(c_{qs}(A)\right)\subseteq c_{q's}\left(f(A)\right)$.

Theorem 3.1.25 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is q_S- continuous on $\underset{i\in I}{\oplus}X_i$ if and only if $c_{qs}\left(f^{-1}(B)\right)\subseteq f^{-1}\left(c_{q's}(B)\right)$ for every $B\subset\underset{j\in J}{\oplus}Y_j$.

Proof If f is q_S- continuous and $B\subseteq\underset{j\in J}{\oplus}Y_j$, We have $c_{q's}(B)$ is q'_S- closed in $\underset{j\in J}{\oplus}Y_j$ and hence $f^{-1}\left(c_{q's}(B)\right)$ is q_S- closed in $\underset{i\in I}{\oplus}X_i$. Therefore
$c_{qs}\left[f^{-1}\left(c_{q's}(B)\right)\right]=f^{-1}\left(c_{q's}(B)\right)$. Since $B\subseteq c_{q's}(B)$ we have $f^{-1}(B)\subseteq f^{-1}\left(c_{q's}(B)\right)$.
Therefore $c_{qs}f^{-1}(B)\subseteq c_{qs}\left(f^{-1}\left(c_{q's}(B)\right)\right)=f^{-1}\left(c_{q's}(B)\right)$. That is $c_{qs}f^{-1}(B)\subseteq f^{-1}\left(c_{q's}(B)\right)$.

Conversely, let $c_{qs}f^{\ 1}(B)\subseteq f^{\ 1}\left(c_{q's}(B)\right)$ for $B\subseteq\underset{j\varepsilon J}{\oplus}Y_j$. Let B be q'_S- closed in $\underset{j\in J}{\oplus}Y_j$ then $c_{q's}(B)=B$. By assumption, $c_{qs}\left(f^{-1}(B)\right)\subseteq f^{-1}(B)$. But $f^{-1}(B)\subseteq c_{qs}\left(f^{-1}(B)\right)$

therefore $c_{qs}\left(f^{-1}(B)\right)=f^{-1}(B)$. That is, $f^{-1}(B)$ is q_S- closed in $\underset{i\in I}{\oplus}X_i$.By Theorem 3.1.21 we have f in q_S- continuous on $\underset{i\in I}{\oplus}X_i$. #.

The following theorem establishes a criteria for q_S- continuous mapping of inverse image of q_S- interior of a subset of $\underset{j\in J}{\oplus}Y_j$.

Theorem 3.1.26 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is q_S - continuous on $\underset{i\in I}{\oplus}X_i$ if and only if $f^{-1}\left(i_{q's}(B)\right)=i_{qs}\left(f^{-1}(B)\right)$ for every subset B of $\underset{j\in J}{\oplus}Y_j$.

Proof Let f be q_S- continuous mapping and $B\subseteq\underset{j\in J}{\oplus}Y_j$.Then $i_{q's}(B)$is q'_S- open in $\underset{j\in J}{\oplus}Y_j$.Therefore $f^{-1}\left(i_{q's}(B)\right)$ is q_S- open in $\underset{i\in I}{\oplus}X_i$.That is,
$f^{-1}\left(i_{q's}(B)\right)=i_{qs}\left(f^{-1}\left(i_{q's}(B)\right)\right)$.Also, $i_{q's}(B)=B$ implies that $f^{-1}\left(i_{q's}(B)\right)=\left(f^{-1}(B)\right)$.
Therefore, $i_{qs}\left[f^{-1}\left(i_{q's}(B)\right)\right]=i_{qs}f^{-1}(B)$.That is $f^{-1}\left(i_{q's}(B)\right)=i_{qs}\left(f^{-1}(B)\right)$.

Conversely, let $f^{-1}\left(i_{q's}(B)\right)=i_{qs}\left(f^{-1}(B)\right)$.for every subset B of $\underset{j\in J}{\oplus}Y_j$. If B is q'_S- open in $\underset{j\in J}{\oplus}Y_j$ then $i_{q's}(B)=B$. Also, $f^{-1}\left(i_{q's}(B)\right)=f^{-1}(B)$. By assumption we have, $f^{-1}(B)=i_{qs}\left(f^{-1}(B)\right)$. Thus $f^{-1}(B)$ is q_S- open in $\underset{i\in I}{\oplus}X_i$ for every q'_S- open set B in $\underset{j\in J}{\oplus}Y_j$. Therefore f is q_S- continuous by Definition 3.1.20. #.

The following theorem establishes a criteria for q_S- continuous mapping in term of inverse image of q_S -kernel of a subset of $\underset{j\in J}{\oplus}Y_j$.

Theorem 3.1.27 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is q_S - continuous on $\underset{i\in I}{\oplus}X_i$ if and only if $f^{-1}\left(k_{qs}(B)\right)\subseteq k_{qs}\left(f^{-1}(B)\right)$ for every subset $B\subseteq\underset{j\in J}{\oplus}Y_j$.

Proof Let f be q_S- continuous mapping and $B\subseteq\underset{j\in J}{\oplus}Y_j$.Then $k_{q's}(B)$ is q'_S- open in $\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$.Therefore $f^{-1}\left(k_{qs}(B)\right)$ is q_S-open in $\left(\underset{i\in I}{\oplus}X_i,q'_S\right)$.That is,
$f^{-1}\left(k_{q's}(B)\right)=f^{-1}(B)\subseteq k_{qs}\left(f^{-1}(B)\right)$ implies that $f^{-1}\left(k_{q's}(B)\right)\subseteq k_{qs}\left(f^{-1}(B)\right)$. #.

In the following theorem we establishes that q_S- continuous maps in term of the arbitrary union of $c_{q's}$ of $f(A)$ and $Y_j, j\in J$.

Theorem 3.1.28 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ be q_S- continuous on $\underset{i\notin I}{\oplus}X_i$ if and only if $f\left(c_{qs}(A)\right)=\underset{j\in J}{\cup}c_{q's}\left(f(A)\cap Y_j\right)$

Proof : Let f be q_S - continuous and $A \subseteq \underset{i\in I}{\oplus} X_i$. So $f(A) \subseteq \underset{j\in J}{\oplus} Y_j$, By Theorem 3.1.22, we have, $f\left(c_{qs}(A)\right) \subseteq c_{qs}\left(f(A)\right) \subseteq \underset{j\in J}{\cup} c_{q's}\left(f(A) \cap Y_j\right)$.Hence $f\left(c_{qs}(A)\right) = \underset{j\in J}{\cup} c_{q's}\left(f(A) \cap Y_j\right)$.

Conversely, let $f\left(c_{qs}(A)\right) = \underset{j\in J}{\cup} c_{q's}\left(f(A) \cap Y_j\right)$ for $A \subseteq \underset{i\in I}{\oplus} X_i$. Let F is q'_S - closed in $\underset{j\in J}{\oplus} Y_j$, we shall prove that $f^{-1}(F)$ is q_S - closed in $\underset{i\in I}{\oplus} X_i$.
By assumption $f\left(c_{qs}\left(f^{-1}(F)\right)\right) \subseteq \underset{j\in J}{\cup} c_{q's}\left(f\left(f^{-1}(F)\right) \cap Y_j\right). \subseteq \underset{j\in J}{\cup} c_{q's}\left(f \cap Y_j\right) \subseteq F$.
Hence$\left(c_{qs}\left(f^{-1}(F)\right)\right) \subseteq \left(f^{-1}(F)\right)$, but $f^{-1}(F) \subseteq c_{qs}\left(f^{-1}(F)\right)$. So $c_{qs}\left(f^{-1}(F)\right) = f^{-1}(F)$.
Thus, $f^{-1}(F)$is q_S - closed in $\underset{i\in I}{\oplus} X_i$. By Theorem 3.1.21, we obtain f is q_S - continuous. #.

Theorem 3.1.29 A mapping $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ is q_S-continuous if and only if $c_{qs}\left(f^{-1}(B)\right) \subseteq \underset{j\in J}{\cup}\left(c_{q's} f^{-1}\left(B \cap Y_j\right)\right)$ for every subset B of $\underset{j\in J}{\oplus} Y_j$.

Proof Since $\underset{j\in J}{\cup} f^{-1}\left(c_{q'_s}\left(B \cap Y_j\right)\right) \subseteq f^{-1}\left(c_{q'_s}(B)\right)$ for $Y_j \subseteq \underset{j\in J}{\oplus} Y_j$, by Theorem 3.1.25 we obtain $c_{q_s}\left(f^{-1}(B)\right) \subseteq \underset{j\in J}{\cup} f^{-1}\left(c_{q'_s}\left(B \cap Y_j\right)\right)$. #.

Theorem 3.1.30 A mapping $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ is q_S - continuous on $\underset{i\in I}{\oplus} X_i$ if and only if $f|_{X_i}: X_i \to Y_j$ is q_i - continuous on $\underset{i\in I}{\oplus} X_i$.

Proof Let f be q_S - continuous mapping on $\underset{i\in I}{\oplus} X_i$ and $X_i \subseteq \underset{i\in I}{\oplus} X_i$. We have $f(X_i) \subseteq \underset{j\in J}{\oplus} Y_j$, *let* $f(X_i) = Y_j$, therefore f is a mapping of X_i into Y_j. So we obtain, $f|_{X_i}: X_i \to Y_j$ is q_i - continuous mapping on X_i.

Conversely, let $f|_{X_i}: X_i \to Y_j$ be q_i -continuous mapping. Clearly $\underset{i\in I}{\cup} f_i = f: \underset{i\in I}{\oplus} X_i \to \underset{j\in J}{\oplus} Y_j$ be q_S - continuous mapping.

Note 3.1.31 By Theorem 3.1.30, we can say, if a mapping f is q_S - continuous on $\underset{i\in I}{\oplus} X_i$, there exist a mapping $f|_{X_i}: X_i \to Y_j$ is q_i - continuous on X_i.

Definition 3.1.32 Let $\left(\underset{i\in I}{\oplus} X_i, q_S\right)$be a q_S - space and $A \subseteq \underset{i\in I}{\oplus} X_i$. is said to be q_S - dense in $\underset{i\in I}{\oplus} X_i$ if $c_{qs}(A) = \underset{i\in I}{\oplus} X_i$.

Example 3.1.33 Let
$I = \{1,2\}, X_1 = \{a,b\}, X_2 = \{c,d\}, q_1 = \{\varnothing, \{a\}, \{a,b\}\}, q_2 = \{\varnothing, \{c\}, \{c,d\}\}$.
We have $\underset{i\in I}{\oplus} X_i = \{a,b,c,d\}$ and $q_S - C\left(\underset{i\in I}{\oplus} X_i\right)$ are $\{\{b,c\}, \{b\}, \{c\}, \{a,b,c,d\}, \varnothing\}$

Let $A=\{a,b,c\}$, we have $c_{qs}(A)=\underset{i\in I}{\oplus}X_i$, hence $A=\{a,b,c\}$ is q_S - dense in $\underset{i\in I}{\oplus}X_i$.

Note 3.1.34 By Definition 3.1.32, and Example 3.1.33 we see that the q_s - dense in $\underset{i\in I}{\oplus}X_i$ if $\varnothing\in q_S-O\left(\underset{i\in I}{\oplus}X_i\right)$. That is, $\varnothing\in q_i-O(X_i)$. for every $i\in I$.

Theorem 3.1.35 Let a mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ be a q_S - continuous and onto. If A is q_S - dense in $\underset{i\in I}{\oplus}X_i$ then $f(A)$ is q'_S - dense in $\underset{j\in J}{\oplus}Y_i$.

Proof Since A is q_S - dense in $\underset{i\subset I}{\oplus}X_i$, we have $c_{qs}(A)=\underset{i\subset I}{\oplus}X_i$. Then $f\left(c_{qs}(A)\right)=f:\left(\underset{i\in I}{\oplus}X_i\right)=\underset{j\in J}{\oplus}Y_j$ since f is onto. Since f is q_S -continuous on $\underset{i\in I}{\oplus}X_i$, $f\left(c_{qs}(A)\right)\subseteq f\left(c_{q's}(A)\right)$. Therefore $\underset{j\in J}{\oplus}Y_i,\subseteq c_{q's}(f(A))$. But $c_{q's}(f(A))\subseteq\underset{j\in J}{\oplus}Y_i$. Therefore, $c_{q's}(f(A))=\underset{j\in J}{\oplus}Y_i$, that is $f(A)$ be q'_S - dense in $\underset{j\in J}{\oplus}Y_i$. #.

3.2 q_S - open and q_S - closed maps.

Definition 3.2.36 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is a q_S - open map if the image of every q_S - open set in $\underset{i\in I}{\oplus}X_i$ is q'_S - open in $\underset{j\in J}{\oplus}Y_j$. The mapping f is said to be a q_S -closed map if the image of every q_S - closed sets in $\underset{i\subset I}{\oplus}X_i$ is q'_S -closed in $\underset{j\subset J}{\oplus}Y_j$.

Example 3.2.37 Let $I=\{1,2\},X_1=\{a,b\},X_2=\{c,d\},q_1=\{\{a\}\},q_2=\{\{c\},\{c,d\}\}$ and $J=\{3,4\},Y_3=\{e,g\},Y_4=\{h,k\},q_3=\{\{e\}\},q_4=\{\{h\},\{h,k\}\}$. We have $\underset{i\in I}{\oplus}X_i=\{a,b,c,d\}$. $\underset{j\in J}{\oplus}Y_j=\{e,g,h,k\}$, $q_S=\{\{a,c\},\{a,c,d\}\}$ and $q'_S=\{\{e,h\},\{e,h,k\}\}$.

Let $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ defined by $f(a)=e,f(b)=g,f(c)=h,f(d)=k$. We find that the image of every q_S - open in $\underset{i\in I}{\oplus}X_i$ is q'_S - open in $\underset{j\in J}{\oplus}Y_j$. Therefore, f is a q_S - open map.

Theorem 3.2.38 A mapping $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is a q_S - closed map if and only if $c_{q's}(f(A))\subseteq f\left(c_{qs}(A)\right)$ for every $A\subseteq\underset{i\in I}{\oplus}X_i$.

Proof If f is q_S - closed map then $f\left(c_{qs}(A)\right)$ is q_S - closed in $\underset{j\in J}{\oplus}Y_i$, since $c_{qs}(A)$ is q_S - closed in $\underset{i\in I}{\oplus}X_i$. Since $A\subseteq c_{qs}(A),f(A)\subseteq f\left(c_{qs}(A)\right)$. Thus $f\left(c_{qs}(A)\right)$ is q'_S - closed set containing $f(A)$. Therefore, $c_{q's}(f(A))\subseteq f\left(c_{qs}(A)\right)$.

Conversely, if $c_{q's}\left(f\left(A\right)\right)\subseteq f\left(c_{qs}\left(A\right)\right)$ for every $A\subseteq\bigoplus_{i\in I}X_i$. Let F is q_S' - closed in $\bigoplus_{i\in I}X_i$;, then $c_{qs}\left(F\right)=F$ and hence $f\left(F\right)\subseteq c_{q's}f\left(F\right)\subseteq f\left(c_{qs}\left(F\right)\right)=f\left(F\right)$.Thus, $f\left(F\right)=c_{qs}\left(f\left(F\right)\right)$.That is, $f\left(F\right)$ is q_S'-closed in $\bigoplus_{j\in J}Y_j$ .Therefore, f is a q_S-closed map. #.

Theorem 3.2.39 A mapping $f:\left(\bigoplus_{i\in I}X_i,q_S\right)\rightarrow\left(\bigoplus_{j\in J}Y_j,q_S'\right)$ is q_S - open map if and only if $f\left(\mathrm{i}_{qs}\left(A\right)\right)\subseteq\mathrm{i}_{q's}\left(f\left(A\right)\right)$ for every $A\subseteq\bigoplus_{i\in I}X_i$.

Proof Let f be q_S - open and let $A\subseteq\bigoplus_{i\in I}X_i$. We know that $f\left(\mathrm{i}_{qs}\left(A\right)\right)$ is q_S' - open set. Since f is a q_S-open mapping, $f\left(\mathrm{i}_{qs}\left(A\right)\right)$ is q_S'-open Since $\mathrm{i}_{qs}\left(A\right)\subseteq A$, we have $f\left(\mathrm{i}_{qs}\left(A\right)\right)\subseteq f\left(A\right)$, $\mathrm{i}_{q's}f\left(\mathrm{i}_{qs}\left(A\right)\right)\subseteq\mathrm{i}_{q's}f\left(A\right)$.Since $f\left(\mathrm{i}_{qs}\left(A\right)\right)$ is q_S'-open.We have $f\left(\mathrm{i}_{qs}\left(A\right)\right)\subseteq\mathrm{i}_{q's}\left(f\left(A\right)\right)$.

Conversely, let $f\left(\mathrm{i}_{qs}\left(A\right)\right)\subseteq\mathrm{i}_{q's}\left(f\left(A\right)\right)$, $A\subseteq\bigoplus_{i\in I}X_i$ and let G be any q_S - open set, so that $\mathrm{i}_{qs}G=G$. Then $f\left(G\right)=f\left(\mathrm{i}_{qs}\left(G\right)\right)\subseteq\mathrm{i}_{q's}f\left(G\right)$(by hypothesis). But $\mathrm{i}_{q's}f\left(G\right)\subseteq f\left(G\right)$. Hence $f\left(G\right)=\mathrm{i}_{q's}\left(f\left(G\right)\right)$. Therefore $f\left(G\right)$ is q_S' - open. #.

3.3 q_S - homeomorphism

Definition 3.3.40 Let $\left(\bigoplus_{i\in I}X_i,q_S\right)$ and $\left(\bigoplus_{j\in J}Y_j,q_S'\right)$ be q_S - space and let f be a mapping of $\bigoplus_{i\in I}X_i$ into $\bigoplus_{j\in J}Y_j$. Then f in said to be q_S - homeomorphism if and only if

1. f is one-to-one and onto,
2. f is q_S - continuous,
3. f^{-1} is q_S - continuous.

Example 3.3.41 By Example 3.2.37, we have f is q_S - homeomorphism.

Theorem 3.3.42 Let $f:\left(\bigoplus_{i\in I}X_i,q_S\right)\rightarrow\left(\bigoplus_{j\in J}Y_j,q_S'\right)$ be an identity map then f is q_S - homeomorphism.

Proof Let f is identity map given

$$f\left(x\right)=x\quad\forall x\in\bigoplus_{i\in I}X_i.$$

Let $G\subseteq\bigoplus_{i\in I}X_i$ be arbitrary q_S - open set. Then

$$\begin{aligned}f^{-1}\left(G\right)&=\left\{x\in\bigoplus_{i\in I}X_i:f\left(x\right)\in G\right\}.\\&=\left\{x\in\bigoplus_{i\in I}X_i:f\left(x\right)\in G\right\}.\\&=G.\end{aligned}$$

So that $f^{-1}(G)$ is q_S - open in $\underset{i\in I}{\oplus} X_i$. Therefore f is q_S - continuous map.

Let $G \in \underset{i\in I}{\oplus} X_i$ be arbitrary q_S - open set. Moreover f is one-one, onto. Hence f is q_S - homeomorphism. #.

Theorem 3.3.43 A mapping $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ be q_S - homeomorphism if and only if $f\Big|_{X_i}:(X_i, q_S) \to (Y_j, q'_S)$ is q_i - homeomorphism.

Proof Since f is one-one, onto so that $f\Big|_{X_i}$ is one - one, onto also. By Theorem 3.1.30 we have $f\Big|_{X_i}$ is q_i - homeomorphism. #.

Theorem 3.3.44 Let $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ be one - one and onto map. Then the following statements are equivalent:

i. f is q_S - open and q_S - continuous,

ii. f is a q_S - homeomorphism,

iii. f is q_S - closed and continuous.

Proof i. $\Rightarrow$ii. : Let f be q_S - open and q_S - continuous. To prove that f is a q_S - homeomorphism, We have to prove that

1. f is one-one onto,
2. f is q_S - continuous,
3. f^{-1} is q'_S - continuous.

By assumption 1. and 2. at once follow $f^{-1}:\left(\underset{j\in J}{\oplus} Y_j, q'_S\right) \to \left(\underset{i\in I}{\oplus} X_i, q_S\right)$. Let $G \in q_S$, then $\left(f^{-1}\right)^{-1}(G) = f(G)$. Since f is q_S - open, hence $G \in q_S \Rightarrow f(G) \in q'_S$. Thus, it given any $G \in q_S$, we can show that $\left(f^{-1}\right)^{-1}(G) \in q_S$. It follows that f^{-1} is q'_S - continuous. Hence the result 3. To prove ii. $\Rightarrow$iii. : Let f be a q_S - homeomorphism. To prove that f is q_S - closed and q_S - continuous since f is q_S - homeomorphism, we have f is q_S - continuous.

Let F be a q_S - closed subset of $\underset{i\in I}{\oplus} X_i$ so that $\underset{i\in I}{\oplus} X_i - F \in q_S$. Since, f is q_S - homeomorphism, we have $f^{-1}:\left(\underset{j\in J}{\oplus} Y_j, q'_S\right) \to \left(\underset{i\in I}{\oplus} X_i, q_S\right)$ is q_S - continuous

where $\underset{i\in I}{\oplus} X_i - F \in q_S \Rightarrow \left(f^{-1}\right)^{-1}\left(\underset{i\in I}{\oplus} X_i - F\right) = f\left(\underset{i\in I}{\oplus} X_i - F\right) \in q'_S$.

$\Rightarrow f\left(\underset{i\in I}{\oplus} X_i\right) - f(F) \in q'_S$.

$\Rightarrow \underset{i\in j}{\oplus} Y_i - f(F) \in q'_S$.

$\Rightarrow f(F)$ is q'_S - closed.

Thus $F \subseteq \underset{i\in I}{\oplus} X_i$ is q_S - closed $\Rightarrow f(F)$ is q'_S - closed.This proves that f is q_S - closed map.

To prove iii. $\Rightarrow$i. Let f be a q_S - closed and q_S - continuous map. To prove that f is q_S - open and q_S - continuous. By assumption, f is q_S - continuous. Let G be q_S - open subset of $\underset{i\in I}{\oplus} X_i$.That is $\underset{i\in I}{\oplus} X_i$ - G is a q_S - closed subset of $\underset{i\in I}{\oplus} X_i$. Consequently $f\left(\underset{i\in I}{\oplus} X_i - G\right)$ is q'_S - closed subset of $\underset{j\in J}{\oplus} Y_j$. For f is given to be q_S - closed

Now $f\left(\underset{i\in I}{\oplus} X_i - G\right) = f\left(\underset{i\in I}{\oplus} X_i\right) - f(G)$.

$= \underset{j\in J}{\oplus} Y_j - f(G)$.

Hence, $\underset{j\in J}{\oplus} Y_j - f(G)$ is q'_S - closed, so that $f(G)$ is q'_S open. Thus $G \in \underset{i\in I}{\oplus} X_i$ is q_S - open, we obtain $f(G)$ is q'_S - open. Therefore f is q_S - open. #.

Theorem 3.3.45 Let $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ be one-to-one, onto. Then f is q_S - homeomorphism if and only if $f\left(c_{qs}(A)\right) = c_{q's}\left(f(A)\right)$ for every subset A of $\underset{i\in I}{\oplus} X_i$.

Proof : If f is q_S - homeomorphism, f is q_S - continuous and q_S - closed. If $A \subseteq \underset{i\in I}{\oplus} X_i$, $f\left(c_{qs}(A)\right) \subseteq c_{q's}\left(f(A)\right)$ since f is q_S - continuous. Since $c_{qs}(A)$ is q_S -closed in $\underset{i\in I}{\oplus} X_i$ and f is q_S -closed, $f\left(c_{qs}(A)\right)$ is q'_S -closed in $\underset{i\in j}{\oplus} Y_i$ and $c_{q's}\left(f(A)\right) \subseteq c_{q's}\left(f\left(\left(c_{q's}A\right)\right)\right) = f\left(c_{q's}(A)\right)$.Therefore, $c_{q's}\left(f(A)\right) \subseteq f\left(c_{q's}(A)\right)$. Thus $f\left(c_{qs}(A)\right) = c_{q's}\left(f(A)\right)$.

Conversely if $f\left(c_{qs}(A)\right) = c_{q's}\left(f(A)\right)$ for every subset A of $\underset{i\in I}{\oplus} X_i$, then f is q_S - continuous. If A is q_S - closed in $\underset{i\in I}{\oplus} X_i, c_{qs}(A) = A$ which implies $f\left(c_{qs}(A)\right) = f(A)$. Therefore $c_{q's}\left(f(A)\right) = f(A)$. Thus, $f(A)$ is q'_S - closed in $\underset{i\in j}{\oplus} Y_i$ for every q_S - closed set A in $\underset{i\in I}{\oplus} X_i$. That is f is q_S - closed. Also, f is q_S - continuous. Therefore f is a q_S - homeomorphism. #.

Theorem 3.3.46 one-to-one onto map $f:\left(\underset{i\in I}{\oplus} X_i, q_S\right) \to \left(\underset{j\in J}{\oplus} Y_j, q'_S\right)$ is a q_S - homeomorphism if and only if $f\left(\mathrm{i}_{qs}(A)\right) = \mathrm{i}_{q's}\left(f(A)\right) \forall A \subseteq \underset{i\in I}{\oplus} X_i$.

Proof Suppose $f:\left(\underset{i\in I}{\oplus}X_i,q_S\right)\to\left(\underset{j\in J}{\oplus}Y_j,q'_S\right)$ is one-to-one onto map. Also suppose that $A\subseteq\underset{i\in I}{\oplus}X_i$ is arbitrary.

i Suppose f is a q_S- homeomorphism. To prove $f\left(\mathrm{i}_{qs}(A)\right)=\mathrm{i}_{q's}\left(f(A)\right)$.

f is a q_S- homeomorphism $\Rightarrow f$ is q_S- continuous.

and $f^{-1}=g$ (say) is q'_S- continuous,

$f^{-1}=g\Rightarrow f=g^{-1}$.

Let $B=f(A)$, then $B\subseteq\underset{j\subset J}{\oplus}Y_j$.

$B\subseteq\underset{j\in J}{\oplus}Y_j$, f is q_S- continuous $\Rightarrow f^{-1}\left(\mathrm{i}_{q's}(B)\right)\subseteq\mathrm{i}_{qs}\left(f^{-1}(B)\right)$.

$\Rightarrow f^{-1}\left(\mathrm{i}_{q's}f(A)\right)\subseteq\mathrm{i}_{qs}\left(f^{-1}\left(f(A)\right)\right)$.

$\Rightarrow f^{-1}\left(\mathrm{i}_{q's}f(A)\right)\subseteq f^{-1}\left(f\left(\mathrm{i}_{qs}(A)\right)\right)$.

$\Rightarrow \mathrm{i}_{q's}\left(f(A)\right)\subseteq f\left(\mathrm{i}_{qs}(A)\right)$. .... (1)

Similarly, $\mathrm{g}:\underset{j\in J}{\oplus}Y_j\to\underset{i\in I}{\oplus}X_i$ is q'_S- continuous, $B\subseteq\underset{j\in J}{\oplus}Y_j$.

$\Rightarrow \mathrm{i}_{qs}\left(g(B)\right)\subseteq g\left(\mathrm{i}_{q's}(B)\right)$.

$\Rightarrow \mathrm{i}_{qs}\left(f^{-1}(A)\right)\subseteq f^{-1}\left(\mathrm{i}_{q's}\left(f(A)\right)\right)$.

$\Rightarrow f^{-1}\left(\mathrm{i}_{qs}f(A)\right)\subseteq\mathrm{i}_{q's}\left(f^{-1}\left(f(A)\right)\right)$.

$\Rightarrow f\left(\mathrm{i}_{qs}(A)\right)\subseteq\mathrm{i}_{q's}\left(f(A)\right)$. .... (2)

Combining (1) and (2), We get the required result.

ii. Conversely, Suppose that $\mathrm{i}_{q's}\left(f(A)\right)=f\left(\mathrm{i}_{qs}(A)\right)$.To prove that f is q_s- homeomorphism, we have to show that f is one-one, onto, f is q_s- continuous and f^{-1} is q_s -continuous. Follows from our initial assumption, we have

$\mathrm{i}_{q's}\left(f(A)\right)\subseteq f\left(\mathrm{i}_{qs}(A)\right)$, ... (3)

and $f\left(\mathrm{i}_{qs}(A)\right)\subseteq\mathrm{i}_{q's}\left(f(A)\right)$. .... (4)

Let $f^{-1}=g$ so that $g^{-1}=f$. By (4) $\Rightarrow g^{-1}\left(\mathrm{i}_{qs}(A)\right)\subseteq\mathrm{i}_{q's}\left[f^{-1}(A)\right]$.

$\Rightarrow g$ is q'_S- continuous by Theorem 3.1.22.

$\Rightarrow f^{-1}$ is q'_S - continuous. #.

Conclusion

The purpose of this paper was to study the continuity on the quasi generalized weak sum space (briefly, q_s - space). As the result we found that, we can study this continuity via q_s - continuous function and discuss some of its properties in terms of q_s - open, q_s - closed , q_s -Interior (i_{qs}) , q_s - closure (c_{qs}) and q_s - kernel (k_{qs}). Furthermore, we discussed some properties in terms of the arbitrary union of q_i - closure (q_i - kernel) of intersection for any set A and X_i. In addition, we discussed q_s - open, q_s - closed mapping and also q_s - homeomorphism. In the future we will generalize continuity on q_s - space.

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